MATH3210 - SPRING 2024 - SECTION 001

HOMEWORK 1 - SOLUTIONS

Problem 1. In the following, A is a set of real numbers. Negate the following statements:

(1) There exists $x \in A$ such that $x < 0$.
(2) Every $x \in A$ is an integer.
(3) For every $x \in A$, there exists $y \in \mathbb{R}$ such that $yx = 1$.
(4) For every $x, y \in A$, if $x \leq y$, then there exists $z \in A$ such that $x < z < y$.

Solution.

(1) For every $x \in A$, $x \geq 0$.
(2) There exists some $x \in A$ such that x is not an integer.
(3) There exists some $x \in A$ such that for every $y \in \mathbb{R}$, $yx \neq 1$.
(4) There exist $x, y \in A$ such that $x \leq y$, but for every $z \in A$, either $z \leq x$ or $y \leq z$.

□

Problem 2. Recall that a set $R \subset \mathbb{Q}$ is a *Dedekind cut* if it satisfies all 3 of the following properties:

(a) $R \neq \emptyset$.
(b) R is bounded above.
(c) For every $x \in R$ and $y \in \mathbb{Q}$, if $y < x$, then $y \in R$.
(d) For every $x \in R$, there exists $y \in R$ such that $y > x$.

Show that if R and S are Dedekind cuts, then $R + S = \{x + y : x \in R \text{ and } y \in S\}$ is also a Dedekind cut (ie, show properties (a)-(d) for the set $R + S$, assuming (a)-(d) for the sets R and S themselves).

Solution. Assume R and S are Dedekind cuts. To see that $R + S$ is a Dedekind cut, we need to show that it satisfies properties (a)-(d) for $R + S$.

(a) We need to show $R + S \neq \emptyset$. Since R and S are Dedekind cuts, they also satisfy (a). Hence there exists $x \in R$ and $y \in S$. By definition, we conclude that $x + y \in R + S$.

(b) We need to show that there exists a number M such that for every $z \in R + S$, $z \leq M$. Since R and S are Dedekind cuts, they have upper bounds. Let L_1 denote an upper bound of R and L_2 denote an upper bound of S. Define $M = L_1 + L_2$. Then if $z \in R + S$, there exists $x \in R$ and $y \in S$ such that $z = x + y$. It follows that for any such z,

$$z = x + y \leq L_1 + L_2 = M.$$

So M is an upper bound of $R + S$.

(c) We need to show property (c) for $R + S$. We will rename the variables for convenience. That is, we will show that if $z \in R + S$, and $w \in \mathbb{Q}$ satisfies $w < z$, then $w \in R + S$.

Let $z \in R + S$, and $w \in \mathbb{Q}$ satisfy $w < z$. By definition, $z = x + y$ for some $x \in R$ and $y \in S$. Then $w - y < z - y = x$. By property (c) for R, it follows that $w - y \in R$. But then since $y \in S$, we conclude that $w = (w - y) + y \in R + S$. This concludes the proof.

(d) We need to show property (d) for $R + S$. We will again rename variables for convenience. That is, we will show that for every $z \in R + S$, there exists $w \in R + S$ such that $w > z$.

Let $z \in R + S$, so that $z = x + y$ for some $x \in R$ and $y \in S$. Since R and S satisfy property (d), we can find elements $a \in R$ and $b \in S$ such that $a > x$ and $b > y$. Therefore, if $w = a + b$, $w \in R + S$, and $w = a + b > x + y = z$.

□

Problem 3. Prove that if $x, y \in \mathbb{R}$ and $x < y$, then there exists some $q \in \mathbb{Q}$ such that $x < q < y$. You may use the following fact: if $a, b \in \mathbb{R}$ and $b - a > 1$, then there exists $m \in \mathbb{Z}$ such that $a < m < b$.

Solution. Suppose that $x, y \in R$ satisfy $x < y$. Then $y - x > 0$. By the Archimedian property, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < y - x$, and hence

$$1 < n(y - x) = ny - nx.$$

By the fact provided, there exists $m \in \mathbb{Z}$ such that $nx < m < ny$. Dividing the chain of inequalities by n yields

$$x < \frac{m}{n} < y.$$

Since $m, n \in \mathbb{Z}$, $q = \frac{m}{n} \in \mathbb{Q}$. and we have shown the claim. □

Problem 4. Let $S_1, S_2, \ldots$ be bounded subsets of $\mathbb{R}$, and $x_n = \sup S_n$. Show that if $S = \bigcup_{n \in \mathbb{N}} S_n$ is also bounded above, then

$$\sup S = \sup \{x_n : x \in \mathbb{N}\}.$$

Solution. Let $y = \sup S$. We will show that $y = \sup \{x_n : x \in \mathbb{N}\}$. To do so, we must show that y is an upper bound of the set $\{x_m : x \in \mathbb{N}\}$, and that if $z < y$, then there exists an element of the set greater than z. First, we show it is an upper bound. Since $y = \sup S$, y is an upper bound of S. Fix an arbitrary $n \in \mathbb{N}$, and let $w \in S_n$. Since $S_n \subset S$, $w \in S$. Since y is an upper bound of S, $w \leq y$. Hence y is an upper bound of S_n for every n. Since x_n is the *least* upper bound of S_n, $x_n \leq y$. Therefore we have shown that for every n, $x_n \leq y$, and y is an upper bound of $\{x_n : n \in \mathbb{N}\}$.

We now show that if $w < y$. w is not an upper bound of $\{x_n : n \in \mathbb{N}\}$. Since $y = \sup S$, and $w < y$, w is not an upper bound of S. Then there exists some $z \in S$ such that $z > w$. Since S is an infinite union, there must exist some n such that $z \in S_n$. Therefore, w is not an upper bound of S_n. Since x_n is an upper bound of S_n, $w < x_n$. This implies that w cannot be an upper bound of $\{x_n : n \in \mathbb{N}\}$. □